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External–Internal Group Quotient Structure for the Standard Model in Analogy to General Relativity

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In analogy to the class structure $\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3)$ for general relativity with a local Lorentz group as stabilizer and a basic tetrad field for the parametrization, a corresponding class structure $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ is investigated for the standard model with a local hyperisospin group $\mathbf{U}(2)$. The lepton, quark, Higgs, and gauge fields used in the standard model cannot be basic in a coset interpretation; they may to be taken as first-order terms in a flat spacetime, particle-oriented expansion of a basic field (as the analogue to the tetrad) and its products.

1. THE COSET STRUCTURE IN RELATIVITY

Usually, general relativity as the dynamics of a metric for a Lorentz manifold is characterized with concepts from differential geometry. To prepare a comparison of relativity and the standard model from a common coset point of view, I present in this section the well-known (Utiyama, 1956) Lorentz group class structure of relativity in a more algebraically oriented language.

Special relativity distinguishes a Lorentz group $\mathbf{O}(1, 3)$ with its causal order-preserving orthochronous subgroup $\mathbf{SO}^+(1, 3)$ as invariance group of a symmetric[2] pseudometric g with signature (1, 3) on a real 4-dimensional vector space $\mathbb{M} \cong \mathbb{R}^4$ with spacetime translations (Minkowski space)

$$g\colon \quad \mathbb{M} \vee \mathbb{M} \to \mathbb{R}, \qquad \text{sign } g = (1, 3), \qquad g(v, w) = g(w, v)$$

$$\mathbf{O}(1, 3) \ni \Lambda\colon \quad \mathbb{M} \to \mathbb{M} \Leftrightarrow G = g \circ (\Lambda \vee \Lambda)$$

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[2] For a vector space V, the totally symmetric and antisymmetric tensor product subspaces are denoted with $V \vee V$ and $V \wedge V$, resp., in the 2nd tensor power $V \otimes V$, correspondingly higher powers, e.g., $V \vee V \vee V$ and $V \wedge V \wedge V$ in $V \otimes V \otimes V$, etc.

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The inverse metric is used for the dual[3] energy-momentum space $\mathbb{M}^T$

$$g^{-1}\colon\quad \mathbb{M}^T \vee \mathbb{M}^T \to \mathbb{R}, \qquad g^{-1} = g^{-1} \circ (\Lambda \vee \Lambda)^{-1T}$$

on which the contragredient representation Λ^{-1T} acts.

A Lorentz metric induces an isomorphism[4] between translations and energy-momenta

$$g\colon\quad \mathbb{M} \to \mathbb{M}^T, \qquad v \mapsto g(v, \cdot), \qquad g = g^T$$

It defines[5] a linear g-involution (Lorentz 'conjugation') $f \overset{g}{\leftrightarrow} f^g$ for all endomorphisms $f\colon \mathbb{M} \to \mathbb{M}$ of the translations

$$\begin{array}{ccc} \mathbb{M} & \xrightarrow{\;g\;} & \mathbb{M}^T \\ {\scriptstyle f^g}\big\downarrow & & \big\downarrow{\scriptstyle f^T} \\ \mathbb{M} & \xrightarrow[\;g\;]{} & \mathbb{M}^T \end{array}, \qquad f^g = g^{-1} \circ f^T \circ g, \quad f^{gg} = f$$

for all $v, x \in \mathbb{M}$: $g(v, f(w)) = g(f^g(v), w)$.

The g-invariance Lorentz group is defined by g-unitarity[6]

$$\Lambda \in \mathbf{O}(1, 3) \Leftrightarrow \Lambda^g = \Lambda^{-1}$$

The invariance Lorentz Lie algebra[7] is g-antisymmetric and therefore as a vector space isomorphic to the antisymmetric square of the translations

$$l \in \log \mathbf{O}(1, 3) \Leftrightarrow l^g = -l$$

$$\mathbb{R}^{16} \cong \mathbb{M} \otimes \mathbb{M}^T \supset \log \mathbf{O}(1, 3) \cong \mathbb{M} \wedge \mathbb{M} \cong \mathbb{R}^6$$

There is a manifold (symmetric space) $\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3)$ of Lorentz groups in the general linear group of a real 4-dimensional vector space as illustrated by the different invariance groups of the three metric matrices (Bourbaki, 1959; Finkelstein, 1996) in one reference basis of the translations

[3] V^T denotes the algebraic dual with the linear forms for the vector space V, $f^T\colon W^T \to V^T$ is the dual (transposed) linear mapping for $f\colon V \to W$. For finite dimensions, the linear mappings $\{f\colon V \to W\}$ are naturally isomorphic to the tensor product $W \otimes V^T$.

[4] The sloppy notation $g\colon \mathbb{M} \vee \mathbb{M} \to \mathbb{R}$ and $g\colon \mathbb{M} \to \mathbb{M}^T$ with the same symbol $g \in \mathbb{M}^T \vee \mathbb{M}^T$ should not lead to confusion.

[5] All diagrams are commutative.

[6] Any involutive $g^{aa} = g \in G$ antiautomorphism $(gh)^a = h^a g^a$ of a group G defines the associated unitary subgroup $U(G, a) = \{g^a = g^{-1}\}$. The inversion is the canonical antiautomorphism. In the quotient $G/U(G, a)$ the unitary group is the stabilizer (Vilenkin and Klimyk, 1991).

[7] The Lie algebra (Bourbaki, 1989b; O'Raifeartaigh, 1986) of a Lie group G is denoted by log G, which recalls also log $\sim$ lag $\sim$ Lie algebra.

$$g \cong \underset{\substack{g = g^T \\ \text{sign } g = (1,\,3)}}{\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}}, \quad \underset{\substack{\text{(time-space bases)} \\ \text{(Sylvester)}}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}},$$

$$\underset{\substack{\text{(light-space bases)} \\ \text{(Witt)}}}{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}, \quad \underset{\substack{\text{(light bases)} \\ \text{(Finkelstein)}}}{\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}}$$

The manifold $\mathbf{GL}(\mathbb{R}^{1+s})/\mathbf{O}(1, s)$ with $s \geq 1$ space dimensions can be visualized for $s = 1, 2$, by all possible s-dimensional 2-component hyperbolas (hyperboloids) in $\mathbb{R}^{1+s}$.

After the Stern–Gerlach experiment leading to the introduction of the spin operations with half integer $\mathbf{SU}(2)$-quantum numbers, also spacetime has to come with a local 'half-integer' Lorentz structure $\mathbf{SL}(\mathbb{C}^2)$. The tetrad field, introduced by Weyl (1929) as the basic field for general relativity, maps a real 4-dimensional differentiable spacetime manifold $\mathcal{D}$, parametrized with four real coordinates $(x^\mu)_{\mu=0}^3 \in \mathbb{R}^4$, into the real 10-dimensional manifold of metrics. It associates a $\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1,\ 3)$-class representative to each spacetime point

$$h\colon \quad \mathcal{D} \to \mathbf{GL}(\mathbb{R}^4), \qquad x \mapsto h(x)$$

It gives an isomorphism between the tangent space, definable by the derivations der $\mathcal{C}(x) = \mathbb{M}(x) \cong \mathbb{R}^4$ of the differentiable functions at each spacetime point $x \in \mathcal{D}$, and one reference translation space $\mathbb{M}(0) \cong \mathbb{R}^4$ with metric $g(0)$,

$$h(x)\colon \quad \mathbb{M}(x) \to \mathbb{M}(0), \qquad h^{-1T}(x)\colon \quad \mathbb{M}^T(x) \to \mathbb{M}^T(0)$$

Therewith all multilinear[8] structures of $\mathbb{M}(0)$ and $\mathbb{M}(x)$ are bijectively related to each other, e.g., the metric and its invariance group,

$$\begin{array}{ccccccc}
(\mathbb{M} \vee \mathbb{M})(x) & \xrightarrow{g(x)} & \mathbb{R} & & \mathbb{M}(x) & \xrightarrow{\Lambda(x)} & \mathbb{M}(x) \\
\Big\downarrow {\scriptstyle h(x)\vee h(x)} & & \Big\downarrow {\scriptstyle \mathrm{id}_R}, & & \Big\downarrow {\scriptstyle h(x)} & & \Big\downarrow {\scriptstyle h(x)} \\
(\mathbb{M} \vee \mathbb{M})(0) & \xrightarrow[g(0)]{} & \mathbb{R} & & \mathbb{M}(0) & \xrightarrow[\Lambda(0)]{} & \mathbb{M}(0)
\end{array}$$

[8] $(\mathbb{M} \otimes \mathbb{M}^T)(x) = \mathbb{M}(x) \otimes \mathbb{M}^T(x)$ or $(\mathbb{M} \vee \mathbb{M})^T(x) = \mathbb{M}^T(x) \vee \mathbb{M}^T(x)$ etc. are vector subspaces of the local tensor algebra over $(\mathbb{M} \oplus \mathbb{M}^T)(x) = \mathbb{M}(x) \oplus \mathbb{M}^T(x)$.

$$g(x) = g(0) \circ (h \vee h)(x), \qquad \Lambda(x) = h^{-1}(x) \circ \Lambda(0) \circ h(x)$$

With dual ($\mathbb{M}(x)$, $\mathbb{M}^T(x)$) bases, e.g., $\{\partial_\mu, dx^\mu\}$, one obtains as tensor components

$$h(x) \sim h^j_\mu(x) \sim h^T(x), \qquad \mathrm{e}h^{-1}(x) \sim h^\mu_j(x) = \frac{\varepsilon^{\mu\nu\rho\lambda}\varepsilon_{jikl}h^i_\nu h^k_\rho h^l_\lambda}{3!\ \det h}(x) \sim h^{-1T}(x)$$

$$g(0) \sim \eta_{jk}, \qquad g^{-1}(0) \sim \eta^{jk}$$

$$g(x) \sim g_{\mu\nu}(x) = \eta_{jk}h^j_\mu h^k_\nu(x), \qquad g^{-1}(x) \sim g^{\mu\nu}(x)$$

With the Lorentz metric-induced isomorphisms between tangent space and its dual, those relations can be written in the form

$$\begin{array}{ccc} \mathbb{M}(x) & \xrightarrow{g(x)} & \mathbb{M}^T(x) \\ {\scriptstyle g(x)}\big\downarrow & & \big\downarrow{\scriptstyle h^{-1T}(x)} \\ \mathbb{M}(0) & \xrightarrow[g(0)]{} & \mathbb{M}^T(0) \end{array}$$

$$g(0) \circ h(x) \sim \eta_{jk}h^k_\mu(x) = h_{j\mu}(x), \qquad g^{-1}(0) \circ h^{-2T}(x) \sim \eta^{jk}h^\mu_k(x) = h^{\mu j}(x)$$

Because of the invariance of the local metric under the local Lorentz transformations

$$g(x) = g(x) \circ (\Lambda \vee \Lambda)(x) = g(0) \circ (h \vee h)(x) \circ (\Lambda \vee \Lambda)(x)$$

the tetrad field as coset representative is determined up to local Lorentz transformations

$$\Lambda(x) \in \mathbf{O}(1, 3)(x): \quad h(x) \mapsto h(x) \circ \Lambda(x)$$

This Lorentz gauge freedom of the tetrad is made compatible with the translations (local derivations) by using $\mathbf{O}(1, 3)$-gauge field $\mathbb{O}(x)$ as a linear mapping from the translations into the Lorentz Lie algebra log $\mathbf{O}(1, 3)(0)$ of the reference space

$$\mathbb{O}(x): \quad \mathbb{M}(x) \to (\mathbb{M} \otimes \mathbb{M}^T)(0), \qquad \mathbb{O}(x) \sim O^i_{j\mu}(x)$$

Because of the Lorentz invariance of the metric, a gauge field is $g(0)$-antisymmetric

$$O(x): \quad \mathbb{M}(x) \to (\mathbb{M} \wedge \mathbb{M})(0)$$

$$O(x) = g^{-1}(0) \circ \mathbb{O}(x) \sim \eta^{ik}O^i_{k\mu}(x) = O^{ij}_\mu(x) = -O^{ji}_\mu(x)$$

General relativity uses no fundamental $\mathbf{O}(1, 3)$-gauge field, but a ‘composite’ one: The local Lorentz freedom for the tetrad defines the tetrad-induced gauge field $\mathbb{O}(x) = \mathbb{O}(h)(x)$ by using a covariantly constant tetrad

$$\begin{aligned} Dh(x)&: \quad \mathbb{M}(x) \otimes \mathbb{M}(x) \to \mathbb{M}(0) \\ & Dh(h) = \partial h(x) - h \circ \Gamma(x) - \mathbb{O} \circ h(x) = 0 \\ & D_\mu h^i_\nu(x) = \partial_\mu h^i_\nu(x) - h^i_\lambda \Gamma^\lambda_{\mu\nu}(x) - O^i_{\mu j} h^j_\nu(x) = 0 \end{aligned}$$

with a manifold connection $\Gamma(x)$. A covariantly constant tetrad leads with $g(x) = g(0) \circ (h \vee h)(x)$ to a covariantly constant metric

$$\begin{aligned} & Dg(x): \quad \mathbb{M}(x) \otimes (\mathbb{M} \vee \mathbb{M})(x) \to \mathbb{R} \\ & Dh(x) = 0 \Rightarrow Dg(x) = 0 = D_\mu g_{\nu\rho}(x) = \partial_\mu g_{\nu\rho}(x) - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho}(x) - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda}(x) \end{aligned}$$

If the log $\mathbf{GL}(\mathbb{R}^4)$-valued connection is assumed as $g(x)$-symmetric (torsion-free manifold), it is expressible by the tetrad and its derivative

$$\text{if } \Gamma^\lambda_{\mu\nu}(x) = \Gamma^\lambda_{\nu\mu}(x) \Rightarrow \Gamma^\lambda_{\mu\nu}(x) = \frac{g^{\lambda\rho}}{2} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})(x)$$

Therewith, the tetrad induced $\mathbf{O}(1, 3)$-gauge field is determined

$$\begin{aligned} \mathbb{O}(x) &= h^{-1} \circ (\partial h - h \circ \Gamma)(x) \\ O^{ij}_\mu(x) &= h^{\nu i}(\partial_\mu h^j_\nu - h^j_\lambda \Gamma^\lambda_{\mu\nu})(x) \\ &= \tfrac{1}{2} h^{\lambda i} h^{\nu j}(h_{\mu k}\partial_{[\lambda} h^k_{\nu]} + h_{\lambda k}\partial_{[\mu} h^k_{\nu]} - h_{\nu k}\partial_{[\mu} h^k_{\lambda]})(x) \end{aligned}$$

The tetrad induced $\mathbf{O}(1, 3)$-curvature field $R(x)$ and $\mathcal{R}(x)$ relates the antisymmetric square of the tangent space and the local Lie algebra log $\mathbf{O}(1, 3)(x) \cong (\mathbb{M} \wedge \mathbb{M})(x)$ to the antisymmetric square of the reference space and the reference Lie algebra

$$\begin{aligned} R(x)&: \quad (\mathbb{M} \wedge \mathbb{M})(x) \to (\mathbb{M} \wedge \mathbb{M})(0) \\ & R(x) \sim R^{ij}_{\mu\nu}(x) = \partial_{[\mu} O^{ij}_{\nu]}(x) - O^{ik}_{[\mu} \eta_{kl} O^{lj}_{\nu]}(x) \\ \mathcal{R}(x)&: \quad (\mathbb{M} \otimes \mathbb{M}^T)(x) \to (\mathbb{M} \otimes \mathbb{M}^T)(0) \\ & \mathcal{R}(x) = g(0) \circ R \circ g^{-1}(x) \sim R^{j\lambda}_{k\mu}(x) = \eta_{ki} R^{ij}_{\mu\nu} g^{\nu\lambda}(x) \end{aligned}$$

With the tetrad isomorphisms, this can be related to a transformation of the reference Lorentz Lie algebra

$$\begin{aligned} R \circ (h \wedge h)^{-1}(x)&: \quad (\mathbb{M} \wedge \mathbb{M})(0) \to (\mathbb{M} \wedge \mathbb{M})(0) \\ & R \circ (h \wedge h)^{-1}(x) \sim R^{ij}_{\mu\nu} h^\mu_k h^\nu_l(x) \\ \mathcal{R} \circ (h \otimes h^{-1})(x)&: \quad (\mathbb{M} \otimes \mathbb{M}^T)(0) \to (\mathbb{M} \otimes \mathbb{M}^T)(0) \\ & \mathcal{R} \circ (h \otimes h^{-1})(x) \sim R^{ij}_{\mu\nu} h^\mu_k h^\nu_l(x) \end{aligned}$$

The coupling of the curvature $\mathcal{R}$ to the tetrad $h \otimes h^{-1}$ determines the familiar second-order derivative action

$$A(h, \partial h) = \int \det h(x)\, d^4x \operatorname{tr} \mathcal{R} \circ (h \otimes h^{-1})(x)$$

$$\operatorname{tr} \mathcal{R} \circ (h \otimes h^{-1})(x) = R^{ij}_{\mu\nu} h^{\mu}_i h^{\nu}_j(x) = R^{j\lambda}_{k\mu} h^{\mu}_j h^{k}_{\lambda}(x) = \operatorname{tr} R \circ (h \wedge h)^{-1}(x)$$

The integration over the manifold uses the invariant volume element

$$\bigwedge^4 \mathbb{M}^T(x) \to \bigwedge^4 \mathbb{M}^T(0), \qquad d^4x \mapsto \frac{\varepsilon_{jikl} h^j_\mu h^i_\nu h^k_\rho h^l_\lambda}{4!}(x)\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\lambda$$

2. THE LINEAR GROUPS OF RELATIVITY

A Lorentz group $\mathbf{O}(1, 3)$ is a semidirect product $\overline{\times}$ of a reflection group[9] $\mathbb{I}(2) = \{\pm 1\}$, e.g., a time reflection, and its special normal subgroup $\mathbf{SO}(1, 3)$ which by itself is the direct product $\times$ of the spacetime translation reflection group $\{\pm \mathbf{1}_4\} \cong \mathbb{I}(2)$ and its orthochronous group $\mathbf{SO}^+(1, 3)$,

$$\mathbf{O}(1, 3) \cong \mathbb{I}(2) \,\overline{\times}\, \mathbf{SO}(1, 3) \cong \mathbb{I}(2) \,\overline{\times}\, [\mathbb{I}(2) \times \mathbf{SO}^+(1, 3)]$$

The general linear group $g \in \mathbf{GL}(\mathbb{R}^4)$ contains via the modulus of the fourth root of the determinant $|\sqrt[4]{\det g}|$ the abelian dilatation group $\mathbf{D}(\mathbf{1}_4) = \mathbf{1}_4 \exp \mathbb{R}$ as a direct factor with the other factor $\mathbf{UL}(\mathbb{R}^4)$ (unimodular linear group) containing the elements with $|\det g| = 1$,

$$\mathbf{GL}(\mathbb{R}^4) = \mathbf{D}(\mathbf{1}_4) \times \mathbf{UL}(\mathbb{R}^4)$$

$$\mathbf{UL}(\mathbb{R}^4) \cong \mathbb{I}(2) \,\overline{\times}\, \mathbf{SL}(\mathbb{R}^4) \cong \mathbb{I}(2) \,\overline{\times}\, [\mathbb{I}(2) \times \mathbf{SL}_0(\mathbb{R}^4)]$$

$\mathbf{SO}^+(1, 3) = \mathbf{SO}_0(1, 3)$ and $\mathbf{SL}_0(\mathbb{R}^4)$ are the connection components of the group unit in $\mathbf{O}(1, 3)$ and $\mathbf{UL}(\mathbb{R}^4)$, respectively, and the adjoint groups[10] of $\mathbf{SO}(1, 3)$ and $\mathbf{SL}(\mathbb{R}^4)$, respectively.

The tetrad manifold is the product of the dilatation group and the quotient of the connection components of the units

$$\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3) \cong \mathbf{D}(1) \times \mathbf{SL}_0(\mathbb{R}^4)/\mathbf{SO}^+(1, 3)$$

The real nine-dimensional manifold $\mathbf{SL}_0(\mathbb{R}^4)/\mathbf{SO}^+(1, 3)$ is the manifold of nontrivial natural order structures $v \succeq 0$ on the translations $\mathbb{M} \cong \mathbb{R}^4$ as induced by the natural order of the scalars $\mathbb{R}$: A natural translation order $\succeq$

[9] $\mathbb{I}(n) = \{z \in \mathbb{C} | z^n = 1\}$ designates the nth cyclotomic group.
[10] The adjoint group of a group G consists of its classes $G/\text{centr } G$ with respect to the centrum.

has to be characterized by $\mathbb{R}$-multilinear forms, the even-linear symmetric forms characterize the pairs $(\succeq, \preceq)$, consisting of an order and its referse. Only the signature (1, 3)-bilinear forms g define nontrivial order pairs: $v \succeq 0$ or $v \preceq 0 \Leftrightarrow g(v, v) \geq 0$.

The orbit $\{h^{-1}(x) \circ \mathbf{O}(1, 3)(0) \circ h(x) | h(x) \in \mathbf{GL}(\mathbb{R}^4)\}$ of a reference Lorentz group by inner automorphisms with $\mathbf{GL}(\mathbb{R}^4)$-operations does not fill the full group $\mathbf{GL}(\mathbb{R}^4)$ because of the nontrivial centralizer, isomorphic to $\mathbf{GL}(\mathbb{R}) = \mathbf{D}(1) \times \mathbb{I}(2)$.

The equivalence classes irrep $\mathbf{SO}^+(1, 3)$ of the irreducible real finite-dimensional representations of an orthochronous Lorentz group with its simple rank-2 Lie algebra are built by two fundamental representations, the real 4-dimensional Minkowski representation $[1|1]$ (cyclic representation[11]), self-dual with the symmetric signature (1, 3) Lorentz metric g, and the real 6-dimensional adjoint representation $[2|0] \oplus [0|2] \cong [1|1] \wedge [1|1]$, self-dual with two symmetric bilinear forms, the definite metric $g \wedge g$ and the signature (3, 3)-Killing metric[12] $\varepsilon(4)$. Correspondingly, there are two types of real, irreducible [13] finite-dimensional representations, those with equal integer or half-integer 'left' and 'right' spin numbers $J_L = J_R = J = 0, \frac{1}{2}, 1, \ldots$, and those with different 'left' and 'right' spin numbers $J_L \neq J_R$, but integer sum:

$$\begin{aligned} &\text{irrep } \mathbf{SO}^+(1, 3) \\ &\quad = \{[2J|2J] \,|\, 2j = 0, 1, \ldots\} \\ &\qquad \cup \{[2J_L|2J_R] \oplus [2J_R|2J_L] \,|\, 2J_{L,R} = 0, 1, \ldots, J_L \\ &\qquad \neq J_R, J_L + J_R = 0, 1, \ldots\} \end{aligned}$$

The dimensions for the representation spaces are

$$\begin{aligned} J_L = J_R = J: &\quad \dim_{\mathbb{R}}[2J|2J] = (2J + 1)^2 \\ J_R \neq J_R: &\quad \dim_{\mathbb{R}}([2J_L|2J_R] \oplus [2J_R|2J_L]) = 2(2J_L + 1)(2J_R + 1) \end{aligned}$$

All representations are self-dual, i.e., they have an $\mathbf{SO}^+(1, 3)$-invariant bilinear form, symmetric as tensor product of the Lorentz metric.

The equivalence classes of the irreducible, real, finite-dimensional representations (Fulton and Harris, 1991; Helgason, 1978) of the special group $\mathbf{SL}_0(\mathbb{R}^4)$, locally isomorphic to $\mathbf{SO}(3, 3)$, with a simple rank-3 Lie algebra, are built by three fundamental representations, the real 4-dimensional cyclic representations [1, 0, 0] and [0, 0, 1], dual to each other, and the real 6-

[11] A cyclic representation generates by its tensor products all representations (up to equivalence).
[12] The $\mathbb{R}^4$-volume element is a symmetric bilinear form $\varepsilon(4) \sim \varepsilon^{ijkl} = \varepsilon^{klij}$ with signature (3, 3) on $\mathbb{R}^4 \wedge \mathbb{R}^4 \cong \mathbb{R}^6$.
[13] The representations $[2J_L|2J_R] \oplus [2J_R|2J_L]$ are decomposable as complex representations.

dimensional representation $[0, 1\ 0] \cong [1, 0, 0] \wedge [1, 0, 0]$, self-dual with the volume form $\varepsilon(4)$:

$$\text{irrep}\,\mathbf{SL}_0(\mathbb{R}^4) = \{[n_1, n_2, n_3] \big| n_{1,2,3} = 0, 1, \ldots\}$$

$$\dim_{\mathbb{R}}[n_1, n_2, n_3]$$

$$= \frac{(n_1+1)(n_3+1)(n_2+1)(n_1+N_2+2)(n_3+n_2+2)(n_1+n_3+n_2+3)}{2!\ 3!}$$

The three natural numbers in $[n_1, n_2, n_3]$ are the linear combination coefficients of the dominant representation weight from the three fundamental weights. The real 15-dimensional adjoint representation is $[1, 0, 1]$.

The decomposition of the $\mathbf{SL}_0(\mathbb{R}^4)$-representations into $\mathbf{SO}^+(1, 3)$-representations is given for the simplest cases, relevant in relativity, in Table I.

The tangent space of the tetrad (metric) manifold is the quotient of the corresponding Lie algebras

$$\log \mathbf{GL}(\mathbb{R}^4)/\log \mathbf{O}(1, 3) \cong \mathbb{M} \vee \mathbb{M} \cong \mathbb{R}^{10}$$

It carries the irreducible representations $[2, 0, 0]$ of $\mathbf{SL}_0(\mathbb{R}^4)$. The curvature $R_{\mu\nu\kappa\lambda}(x) = R^{ij}_{\mu\nu} h_{i\kappa} h_{j\lambda}(x)$ with its familiar (anti)symmetry properties as traceless element of $(\mathbb{M} \wedge \mathbb{M})(x) \vee (\mathbb{M} \wedge \mathbb{M})(x)$ transforms with the 20-dimensional representation $[0, 2, 0]$, the symmetric Ricci tensor $R_{\mu\lambda}(x) = R_{\mu\nu\kappa\lambda} b^{\nu\kappa}(x)$ with the 10-dimensional $[2, 0, 0]$.

In general, a representation ψ of a group quotient G/U will be defined as a mapping from the classes ψ: $G/U \to V_U \otimes V_G^T$ into the linear mappings ψ_{gU}: $V_G \to V_U$ of two vector spaces with linear representations of the groups involved, $G \to \mathbf{GL}(V_G)$ and $U \to \mathbf{GL}(V_U)$. If the vector spaces are isomorphic, $V_G \cong V_U \cong V$, the mappings $\psi_{gU} \in \mathbf{GL}(V)$ are assumed to be isomorphisms.

The tetrad $h(x)$, $h^{-1}(x) \in \mathbf{GL}(\mathbb{R}^4)$ and the curvature $\mathcal{R}(x) \in \mathbf{GL}(\mathbb{R}^6)$ as representations of the quotient $\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3)$ relate to each other as vector spaces with the fundamental representations of the orthogonal and special group. In general, the $(n - 1)$ fundamental $\mathbf{SL}_0(\mathbb{R}^n)$-representations act on

Table I

$\mathbf{SL}_0(\mathbb{R}^4)$	$[1, 0, 0]$ $[0, 0, 1]$	$[0, 1, 0]$	$[2, 0, 0]$ $[0, 0, 2]$	$[0, 2, 0]$	$[1, 0, 1]$
Dimension	4	$6 = \binom{4}{2}$	$10 = \binom{4+1}{2}$	$20 = \binom{6+1}{2} - 1$	$15 = 4^2 - 1 = \binom{6}{2}$
$\mathbf{SO}^+(1, 30$	$[1 \vert 1]$	$[2 \vert 0] \oplus [0 \vert 2]$	$[0 \vert 0] \oplus [2 \vert 2]$	$[0 \vert 0] \oplus [2 \vert 2] \oplus$ $[4 \vert 0] \oplus [0 \vert 4]$	$[0 \vert 0] \oplus [2 \vert 2] \oplus$ $[2 \vert 0] \oplus [0 \vert 2]$

the $(n-1)$ Grassmann powers $\bigwedge^N \mathbb{R}^n$ for $N = 1, \ldots, n-1$. Therefore the reference Grassmann algebra[14] $\bigwedge \mathbb{M}(0) \cong \mathbb{R}^{16}$ over the translations with the powers $\bigwedge^N \mathbb{M}(0) < \mathbb{R}^{\binom{4}{N}}$ as direct summands and the isomorphic local partners $\bigwedge \mathbb{M}(x)$ are related to each other by the fields in relativity as shown in Table II. The Grassmann degree N is the $\mathbf{D}(1)$-grading, called by Weyl (1973) the 'weight of a tensor density.'

3. THE SCALES FOR RELATIVITY

The rank of the symmetric space $\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3)$ (tetrad or metric manifold) will be defined as the difference $4 - 2$ of the ranks for the 'nominator' and 'denominator' Lie algebra

$$\text{rank}_{\mathbb{R}}\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3) = 2, \qquad \text{rank}_{\mathbb{R}}\mathbf{D}(1) = 1$$

The rank gives the number of invariants for the representations of the manifold: one abelian invariant for $\mathbf{D}(1)$ and one simple invariant for the quotient $\mathbf{SL}_0(\mathbb{R}^4)/\mathbf{SO}^+(1, 3)$. Those invariants can be used as overall normalization and relative space-time normalization, respectively, or as fundamental intrinsic length scale l (Newton's constant) and fundamental velocity scale c,

$$g(x) \cong \frac{l^2}{c}\begin{pmatrix} 1/c & 0 \\ 0 & -c\mathbf{1}_3 \end{pmatrix} = h(l, c)\begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix} h^T(l, c)$$

$$h(l, c) = \begin{pmatrix} l/c & 0 \\ 0 & l\mathbf{1}_3 \end{pmatrix} \qquad \text{with} \quad l, c > 0$$

The abelian invariant is given by the determinant of the tetrad $h(l, c)$ or, in the Lie algebra, by the trace; the simple invariant arises from the 'double trace' as familiar from the Killing form and the quadratic Casimir element for semisimple Lie algebras,

Table II. Lorentz and Special Linear Representation Properties of the Relativity Fields

N	Grassmann power $\bigwedge^n \mathbb{M}$	Field	$\mathbf{SO}^+(1, 3)$	$\mathbf{SL}_0(\mathbb{R}^4)$
0	$\mathbb{R}$	$\text{id}_{\mathbb{R}} \sim 1$	$[0\|0]$	$[0, 0, 0]$
1	$\mathbb{M} \cong \mathbb{R}^4$	$h(x) \sim h^j_\mu(x)$	$[1\|1]$	$[1, 0, 0]$
2	$\mathbb{M} \wedge \mathbb{M} \cong \mathbb{R}^6$	$\mathcal{R}(x) \sim R^{ij}_{\mu\nu}(x)$	$[2\|0] \oplus [0\|2]$	$[0, 1, 0]$
3	$\mathbb{M} \wedge \mathbb{M} \wedge \mathbb{M} \cong \mathbb{R}^4$	$h^{-1}(x) \sim h^\mu_j(x)$	$[1\|1]$	$[0, 0, 1]$
4	$\mathbb{R}$	$\det h(x)$	$[0\|0]$	$[0, 0, 0]$

[14] $\bigwedge \mathbb{M}$ is isomorphic as vector space, not as associative algebra, to the Clifford algebra over $\mathbb{M}$.

$$h(l, c) = \exp l(l, c), \qquad \begin{cases} \det h(l, c) = \exp \operatorname{tr} l(l, c) = l^4/c \\ \exp\sqrt{\dfrac{4 \operatorname{tr} l(l, c) \circ l(l, c) - (\operatorname{tr} l(l, c))^2}{3}} = \dfrac{1}{c} \end{cases}$$

The flat spacetime expansion for general relativity uses the 10-dimensional tangent space of the tetrad manifold. It expands the $\mathbf{GL}(\mathbb{R}^4)$-tetrad with its Lie algebra around a reference Lorentz group $\mathbf{O}(1, 3)$. A tetrad from the unit connection component $\mathbf{GL}_0(\mathbb{R}^4) = \mathbf{D}(\mathbf{1}_4) \times \mathbf{SL}_0(\mathbb{R}^4)$ can be written with an exponent

$$h(x) = \exp l(x), \qquad l(x) \in \log \mathbf{GL}(\mathbb{R}^4)$$

Because of the local invariance, the Lie algebra element $l(x)$ is determined up to gauge translations $l(x) + \log \mathbf{O}(1, 3)(x)$. The flat spacetime expansion is characterized by

$$h(x) = \mathbf{1}_4 + l(x) + \ldots, \qquad h^j_\mu(x) = \delta^k_\mu[\delta^j_k + l^j_k(x) + \ldots]$$

4. THE OPERATION GROUPS OF THE STANDARD MODEL

Before trying an interpretation with coset structures also for the standard model of the electroweak and strong interactions, its relevant operational symmetries will be summarized.

The standard model implements the electroweak and strong interactions as gauge structures, relating the spacetime translations to the internal transformation groups

$$\text{hypercharge: } \mathbf{U}(1), \qquad \text{isospin: } \mathbf{SU}(2), \qquad \text{color: } \mathbf{SU}(3)$$

In the lepton, quark, Higgs, and gauge fields, the internal groups meet with the external transformation groups[15]

$$\text{Lorentz group: } \mathbf{SL}(\mathbb{C}^2), \qquad \text{chirality: } \mathbf{U}(1)$$

The fundamental standard model fields transform internally with irreducible representations $[y]$, $[2T]$, and $[C_1, C_2]$ for hypercharge, isospin, and colour group, respectively, and, externally, with $[c]$ and $[2J_L|2J_R]$ for chirality and Lorentz group, respectively, as given by the quantum numbers in Table

[15] The unspecified name 'Lorentz group' is used for the locally isomorphic real Lie groups $\mathbf{O}(1, 3)$, $\mathbf{SO}(1, 3)$ (special), $\mathbf{SO}^+(1, 3)$ (orthochronous), and $\mathbf{SL}(\mathbb{C}^2)$ (covering). The complex finite-dimensional representations of the real dimension 6, rank 2 simple Lie algebra log $\mathbf{SL}(\mathbb{C}^2)$ are denoted with two natural numbers $[2J_L|2J_R]$ for the linear combination of its dominant weight from the two fundamental weights for the Weyl representations.

III. With respect to the Lorentz group, $[0|0]$ designates scalar fields, $[1|0]$ and $[0|1]$ are left- and right-handed Weyl spinor fields, respectively, and $[1|1]$ designates vector fields. The external and internal multiplicity (singlet, doublet, triplet, quartet, octet, etc.) of the Lorentz group, isospin, and color representations can be computed from the natural numbers $2J_{L,R}$, $2T$, $C_{1,2}$:

$$\dim_{\mathfrak{a}}[2J_L|2J_R] = (2J_L + 1)(2J_R + 1), \qquad 2J_{L,R} = 0, 1, \ldots$$

$$\dim_{\mathfrak{a}}[2T] = 2T + 1, \qquad 2T = 0, 1, \ldots$$

$$\dim_{\mathfrak{a}}[C_1, C_2] = \frac{(C_1 + 1)(C_2 + 1)(C_1 + C_2 + 2)}{2}, \qquad C_{1,2} = 0, 1, \ldots$$

Fields and antifields have reflected quantum numbers

$$\Psi \quad \text{with} \quad [y\|2T; C_1, C_2] \circ [c\|2J_L|2J_R]$$
$$\Psi^* \quad \text{with} \quad [-y\|2T; C_2, C_1] \circ [-c\|2J_R|2J_L]$$

The chirality property $[c]$ will be discussed below in more detail.

The gauge interaction of the fermion fields is effected by the local Lie algebra invariants[16] (current-gauge field products)

$$g_1\mathbf{J}(1)\mathbf{A} + g_2\mathbf{J}(2)\mathbf{B} + g_3\mathbf{J}(3)\mathbf{G}$$

$$\text{for } \mathbf{U}(1): \quad \mathbf{J}(1) = \tfrac{1}{6}[\mathbf{q}^*\mathbf{1}_6\mathbf{q} - 2\mathbf{d}^*\mathbf{1}_3\mathbf{d} - 3\mathbf{l}^*\mathbf{1}_2\mathbf{l} + 4\mathbf{u}^*\mathbf{1}_3\mathbf{u} - 6\mathbf{e}^*\mathbf{e}]$$

$$\text{for } \mathbf{SU}(2): \quad \mathbf{J}(2) = \tfrac{1}{2}[\mathbf{q}^*\overline{\tau} \otimes \mathbf{1}_3\mathbf{q} + \mathbf{l}^*\overline{\tau}\mathbf{l}]$$

$$\text{for } \mathbf{SU}(3): \quad \mathbf{J}(3) = \tfrac{1}{2}[\mathbf{q}^*\mathbf{1}_2 \otimes \overline{\lambda}\mathbf{q} + \mathbf{d}^*\overline{\lambda}\mathbf{d} + \mathbf{u}^*\overline{\lambda}\mathbf{u}]$$

Table III. Quantum Numbers of the Standard Model Fields

Field	Symbol Ψ	**U**(1) $[y]$	**SU**(2) $[2T]$	**SU**(3) $[C_1, C_2]$	**U**(1) $[c]$	**SL**($\mathbb{C}^2$) $[2J_L\|2J_R]$
Left lepton	**l**	$-\frac{1}{2}$	[1]	[0, 0]	$\frac{1}{2}$	$[1\|0]$
Right lepton	**e**	-1	[0]	[0, 0]	$\frac{3}{2}$	$[0\|1]$
Left quark	**q**	$\frac{1}{6}$	[1]	[1, 0]	$-\frac{1}{2}$	$[1\|0]$
Right down quark	**d**	$-\frac{1}{3}$	[0]	[1, 0]	$\frac{1}{2}$	$[0\|1]$
Right up quark	**u**	$\frac{2}{3}$	[0]	[1, 0]	$-\frac{3}{2}$	$[0\|1]$
Higgs	**H**	$-\frac{1}{2}$	[1]	[0, 0]	1	$[0\|0]$
Hypercharge gauge	**A**	0	[0]	[0, 0]	0	$[1\|1]$
Isospin gauge	**B**	0	[2]	[0, 0]	0	$[1\|1]$
Color gauge	**G**	0	[0]	[1, 1]	0	$[1\|1]$

[16] For a Lie algebra representation $\mathfrak{D}: L \to V \otimes V^T$ in the endomorphism algebra of a vector space (L and V finite dimensional) the tensor $\mathfrak{D} \in V \otimes V^T \otimes L^T$ is the associated invariant.

involving as a basis, e.g., the three Pauli and eight Gell-Mann matrices $\vec{\tau} = (\tau^a)_{a=1}^3$ and $\vec{\lambda} = (\lambda^c)_{c=1}^8$, respectively. The coupling constants $g_{1,2,3}^2 > 0$ are the normalizations of the corresponding Lie algebras (Saller, 1998).

At face value, the relevant group seems to be a product of five unrelated direct factors

$$\underbrace{\mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)}_{\text{internal}} \times \underbrace{\mathbf{U}(1) \times \mathbf{SL}(\mathbb{C}^2)}_{\text{external}}$$

A closer look, however, suggests a common origin for all those groups: The three internal factors are related to each other as well as the two external ones and, what is highly interesting, there exists also an internal–external correlation.

In general, a standard model field does not represent faithfully all operations. If a group G is represented, the faithfully represented group is the quotient G/N, consisting of classes with respect to the trivially represented invariant subgroup $N \subseteq G$. To find those groups in the standard model, one has to consider the four central correlations of its operation group (Hucks, 1991; Saller, 1998).

The two internal correlations connect hypercharge with both isospin and color: The colorless fields **l**, **e**, **H**, **A**, and **B** show a (half)integer hypercharge–(half)integer isospin correlation. The isospin-less fields **u**, **d**, and **G** show an analogous $\mathbb{I}(3)$ correlation. Therefore the faithfully represented groups arise from the full unitary groups $\mathbf{U}(n)$ for $n = 2, 3$. $\mathbf{U}(n)$ is a product, not direct, of two normal subgroups with $\mathbb{I}(n)$ as discrete intersection.[17] Its quotient groups are the phase group $\mathbf{U}(\mathbf{1}_n) = \mathbf{1}_n \exp i\mathbb{R}$ and the adjoint group $\mathbf{SU}(n)/\mathbb{I}(n)$ (Table IV),

$$\left.\begin{array}{l} \mathbf{U}(n) = \mathbf{U}(\mathbf{1}_n) \circ \mathbf{SU}(n) \\ \mathbf{U}(\mathbf{1}_n) \cap \mathbf{SU}(n) = \text{centr}\, \mathbf{SU}(n) \cong \mathbb{I}(n) \end{array}\right\} \Rightarrow \mathbf{U}(n) \cong \frac{\underline{\mathbf{U}(1) \times \mathbf{SU}(n)}}{\mathbf{I}(n)}$$

Furthermore, the internal color and isospin properties of the left-handed quark field **q** show that the internal faithfully represented group, defined in $\mathbf{U}(6)$,

Table IV. Internal Operation Groups from $\mathbf{U}(n)$, $n = 2, 3$

Normal subgroup	$\mathbf{U}(1)$	$\mathbf{SU}(n)$
Quotient group	$\mathbf{SU}(n)/\mathbb{I}(n)$	$\mathbf{U}(1)$

[17] The somewhat ambiguous notation $\underline{G_1 \times G_2}/H$ denotes a common normal subgroup $H \subseteq G_1 \cap G_2$ in contrast to, e.g., $G_1 \times G_2/H$.

Table V. Internal Operation Groups from $\mathbf{U}(2 \times 3)$

Normal subgroup	$\mathbf{U}(1)$	$\mathbf{SU}(2)$	$\mathbf{SU}(3)$
Quotient group	$\mathbf{SU}(3) \times \mathbf{SU}(3)/\mathbb{I}(3)$	$\mathbf{U}(3)$	$\mathbf{U}(2)$
Normal subgroup	$\mathbf{U}(2)$	$\mathbf{U}(3)$	$\mathbf{SU}(2) \times \mathbf{SU}(3)$
Quotient group	$\mathbf{SU}(3)/\mathbb{I}(3)$	$\mathbf{SO}(3)$	$\mathbf{U}(1)$

is a product of three normal subgroups with an $\mathbb{I}(2) \times \mathbb{I}(3) \cong \mathbb{I}(6)$ correlation (Table V),

$$\mathbf{U}(2 \times 3) = \mathbf{U}(\mathbf{1}_6) \circ [\mathbf{SU}(2) \otimes \mathbf{1}_3 \times \mathbf{1}_2 \otimes \mathbf{SU}(3)]$$

$$\left.\begin{array}{l} \mathbf{U}(\mathbf{1}_6) \cap [\mathbf{SU}(2) \otimes \mathbf{1}_3] \cong \mathbb{I}(2) \\ \mathbf{U}(\mathbf{1}_6) \cap [\mathbf{q}_2 \otimes \mathbf{SU}(3)] \cong \mathbb{I}(3) \end{array}\right\} \Rightarrow \mathbf{U}(2 \times 3) \cong \frac{\mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)}{\mathbf{I}(2) \times \mathbf{I}(3)}$$

The external correlation is seen in the fact that half-integer spin $J_L + J_R$ comes with half-integer chirality number c and integer $J_L + J_R$ with integer c. Therefore, the faithfully represented external group is the unimodular group $\mathbf{UL}(2) = \{g \in \mathbf{GL}(\mathbb{C}^2) \| \det g| = 1\}$ (phase Lorentz group). Its quotient groups are the phase group (chirality group) and the orthochronous Lorentz group as adjoint group (Table VI),

$$\left.\begin{array}{l} \mathbf{UL}(2) = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SL}(\mathbb{C}^2) \\ \mathbf{U}(\mathbf{1}_2) \cap \mathbf{SL}(\mathbb{C}^2) = \text{centr } \mathbf{SL}(\mathbb{C}^2) \cong \mathbb{I}(2) \end{array}\right\} \Rightarrow \mathbf{UL}(2) \cong \frac{\mathbf{U}(1) \times \mathbf{SL}(\mathbb{C}^2)}{\mathbf{I}(2)}$$

$$\Rightarrow \left\{\begin{array}{l} \mathbf{UL}(2)/\mathbf{SL}(\mathbb{C}^2) \cong \mathbf{U}(1)/\mathbb{I}(2) \cong \mathbf{U}(1) \\ \mathbf{UL}(2)/\mathbf{U}(\mathbf{1}_2) \cong SL(\mathbb{C}^2)/\mathbb{I}(2) \cong \mathbf{SO}^+(1, 3) \end{array}\right.$$

Before discussing the internal–external correlation, the standard model fields will be arranged with respect to the external and internal quotient groups of $\mathbf{UL}(2)$ and $\mathbf{U}(2 \times 3)$, respectively, that they represent faithfully (Table VII). Some entries are missing: First of all, there are no colored Lorentz scalar fields analogous to the Higgs isodoublet. Second, a field of the standard model has nontrivial hypercharge if and only if it has nontrivial chirality. The chirality $\mathbf{U}(1)_{\text{ext}}$ number c is determined from the Yukawa interaction

$$(\mu_e \mathbf{e}^*\mathbf{1} + \mu_u \mathbf{q}^*\mathbf{u} + \mu_d \mathbf{d}^*\mathbf{q})\mathbf{H} + \text{h.c.} \qquad \text{with Yukawa couplings } \mu_{e,u,d} \in \mathbb{R}$$

With an integer $c_\mathbf{H}$ for the Higgs fields, the chiral numbers for the quark

Table VI. External Operation Groups from $\mathbf{UL}(2)$

Normal subgroup	$\mathbf{U}(1)$	$\mathbf{SL}(\mathbb{C}^2)$
Quotient group	$\mathbf{SO}^+(1, 3)$	$\mathbf{U}(1)$

Table VII. Faithfully Represented Homogeneous Groups in the Standard Model

	UL(2)	**U**(1)$_{\text{ext}}$	**SO**$^{+}$(1, 3)
U(2)	**l**	**H**	×
U(1)$_{\text{int}}$	**e**	—	×
U(2 × 3)	**q**	—	×
U(3)	**d**, **u**	—	×
SO(3)	×	×	B
{1}	×	×	A
SU(3)/𝕀(3)	×	×	G

fields **q**, **d**, **u** and for the lepton fields **l**, **e** are given up to integers z_q and z_l in Table VIII.

The choice of the three integers $c_{\mathbf{H}}$, z_l, z_q is not obvious. z_l and z_q will be determined by opposite chirality and hypercharge for the lepton isodoublet field **l** and opposite chirality and threefold hypercharge for the quark isodoublet field **q**,

$$c_{\mathbf{l}} = -y_{\mathbf{l}}, \qquad c_{\mathbf{q}} = -3y_{\mathbf{q}} \Rightarrow z_l, z_q = 0$$

The chirality $c_{\mathbf{H}}$ for the Higgs field is determined in such a way that the hypercharge–chirality combination (fermion number) $f = -c - 2c_{\mathbf{H}}y$, trivial for the Higgs field, gives a ratio 1:3 for quark and lepton fields

$$f_{\mathbf{l}} = 3f_{\mathbf{q}} \Leftrightarrow c_{\mathbf{l}} + 2c_{\mathbf{H}}y_{\mathbf{l}} = 3(c_{\mathbf{q}} + 2c_{\mathbf{H}}y_{\mathbf{q}}) \Rightarrow c_{\mathbf{H}} = 1$$

These conditions will be discussed in Sections 5 and 6.

Both **U**(1)'s, chirality and hypercharge, have to be represented in the only one phase group of a field. The combination of chirality and hypercharge with a trivial value for the Higgs field defines a fermion number group **U**(1) which correlates external and internal **U**(1),

Table VIII. Hypercharge, Chirality, and Fermion Numbers for the Standard Model Fields

	U(1)$_{\text{int}}$ y	**U**(1)$_{\text{ext}}$ c	**U**(1)$_{\text{ext}}$ with $c_{\mathbf{h}} = 1$, $z_{q,l} = 0$	**U**(1)$_{\text{ferm}}$ $f = -c - 2y$
l	$-\frac{1}{2}$	$\frac{1}{2} + z_l$	$\frac{1}{2}$	$\frac{1}{2}$
e	-1	$\frac{1}{2} + z_l + c_{\mathbf{H}}$	$\frac{3}{2}$	$\frac{1}{2}$
q	$\frac{1}{6}$	$-\frac{1}{2} + z_q$	$-\frac{1}{2}$	$\frac{1}{6}$
d	$-\frac{1}{3}$	$-\frac{1}{2} + z_q + c_{\mathbf{H}}$	$\frac{1}{2}$	$\frac{1}{6}$
u	$\frac{2}{3}$	$-\frac{1}{2} + z_q - c_{\mathbf{H}}$	$-\frac{3}{2}$	$\frac{1}{6}$
H	$-\frac{1}{2}$	$c_{\mathbf{H}}$	1	0
A, **B**, G	0	0	0	0

$$\left.\begin{array}{l}\mathbf{U}(1)_{\text{ext}} \subset \mathbf{UL}(2)\\ \mathbf{U}(1)_{\text{int}} \subset \mathbf{U}(2\times 3)\end{array}\right\},\qquad \mathbf{U}(1)_{\text{ferm}} \cong \frac{\mathbf{U}(1)_{\text{ext}} \times \mathbf{U}(1)_{\text{int}}}{\mathbf{U}(1)}$$

$$f = -c - 2y = \begin{cases}\frac{1}{2} & \text{for lepton fields } \mathbf{l}, \mathbf{e}\\ \frac{1}{6} & \text{for quark fields } \mathbf{q}, \mathbf{d}, \mathbf{u}\\ 0 & \text{for boson fields } \mathbf{H}, \mathbf{A}, \mathbf{B}, \mathbf{G}\end{cases}$$

Summarizing the operation groups of the standard model: The external–internal homogeneous symmetry group, faithfully represented with the standard model fields, is a product of five normal subgroups with a fourfold central correlation

$$\frac{\mathbf{U}(2\times 3)\times \mathbf{UL}(2)}{\mathbf{U}(1)} \cong \frac{\mathbf{U}(1)\times \mathbf{SU}(2)\times \mathbf{SU}(3)\times \mathbf{U}(1)\times \mathbf{SL}(\mathbb{C}^2)}{\mathbf{L}(2)\times \mathbf{L}(3)\times \mathbf{U}(1)\times \mathbf{L}(2)}$$

5. SYMMETRIES FOR PARTICLES

One has to make a clear distinction between the operation group (symmetry) for fields and the operation group (symmetry) for particles (Wigner, 1939). Going from the standard model fields for the description of the dynamics to the in- and out-fields for the description of particles, the homogeneous real 18-dimensional Lie group $\mathbf{U}(2\times 3)\times \mathbf{UL}(2)/\mathbf{U}(1)$ with both external and internal operations is dramatically reduced. With color confinement and ground-state frozen electroweak symmetries there remains from the 12-dimensional $\mathbf{U}(2\times 3)$ only a 1-dimensional abelian $\mathbf{U}(1)$ symmetry, faithfully represented by particles with nontrivial electromagnetic charge or fermion number, e.g., by the electron or the neutron. The establishment of a laboratory distinguishes a reference rest system and reduces the 6-dimensional external Lorentz group operations $\mathbf{SL}(\mathbb{C}^2)$ for fields in the case of massive half-integer and integer-spin particles to a faithfully represented 3-dimensional group $\mathbf{SU}(2)$ and $\mathbf{SU}(2)/\mathbb{I}(2) \cong \mathbf{SO}(3)$, respectively. Massless particles represent faithfully only a 1-dimensional polarization subgroup $\mathbf{SO}(2) \cong \mathbf{U}(1) \subset \mathbf{SU}(2)$, which, possibly reflecting the external–internal $\mathbf{U}(1)$ correlation, are all chargeless, e.g., the photon and the neutrinos (Table IX).

6. THE COSET STRUCTURE IN THE STANDARD MODEL

After the coset formulation for relativity in Sections 1–3 and the exposition of the standard model operation groups in Section 4, I come to the main purpose of this paper.

An attempt to characterize the standard model for the electroweak and strong interactions with coset structures and symmetric spaces in analogy to

Table IX. Particles from Standard Fields

Particle	Symbol	**D**(1) mass	**U**(1) ⊂ **SU**(2) spin direction	**U**(1) polarization (helicity)	**U**(1) em charge
Massive electron	$e^{\mp}$	m_e	$+\frac{1}{2}, -\frac{1}{2}$	—	∓ 1
Electron neutrino	$\nu_e, \overline{\nu}_e$	0	—	± 1	0
Charged weak boson	$W^{\pm}$	m_W	$+1, 0, -1$	—	± 1
Neutral weak boson	Z	m_Z	$+1, 0, -1$	—	0
Photon	γ	0	—	± 1	0

relativitry encounters characteristic differences: Relativity is a real theory with orthogonal groups and bilinear forms (metrics), whereas the standard model and quantum theory come in a complex formulation with unitary groups and sesquilinear forms (scalar products, probability amplitudes). The local operation Lorentz group **O**(1, 3) for relativity has no true normal Lie subgroup, whereas the internal standard model operation group **U**(2 × 3) has the normal Lie subgroups **U**(1) (hypercharge), **SU**(2) (isospin), and **SU**(3) (color). The main apparent obstacle for a symmetric space interpretation for the standard model is the color group **SU**(3): It prevents a naive embedding of the internal group **U**(2 × 3) as subgroup of the external phase Lorentz group **UL**(2), as compared to the tetrad manifold quotient structure $\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1,3)$. Therefore, Weinberg's (1967) 'model of leptons' is considered first: There, the colorless group **U**(2) × **UL**(2)/**U**(1) with hyperisospin and phase Lorentz group is represented by the lepton fields **l**, **e**, the hypercharge and isospin gauge fields **A**, **B**, and the Higgs field **H**.

A group **U**(2) (hyperisospin) is the invariance group of a definite scalar product d for a complex 2-dimensional vector space $\mathbb{U} \cong \mathbb{C}^2$

$$d\colon \quad \mathbb{U} \times \mathbb{U} \to \mathbb{C}, \qquad d(v, v) > 0 \Leftrightarrow v \neq 0, \qquad d(v, w) = \overline{d(w, v)}$$

$$\mathbf{U}(2) \ni u\colon \quad \mathbb{U} \to \mathbb{U} \Leftrightarrow d = d \circ (u \times u)$$

A scalar product d for quantum theory is the analogue to a signature (1, 3) metric g of the real translation vector space $\mathbb{M} \cong \mathbb{R}^4$ in relativity with **O**(1, 3) invariance (Section 1).

A scalar product defines a conjugation $f \overset{d}{\to} f^*$ for all linear mappings $f\colon \mathbb{U} \to \mathbb{U}$

$$\text{for all} \quad v, w \in \mathbb{U}\colon \quad d(v, f(w)) = d(f^*(v), w), \qquad f^{**} = f$$

with $u \in \mathbf{U}(2) \Leftrightarrow u^* = u^{-1}$ and $l \in \log \mathbf{U}(2) \Leftrightarrow l = -l^*$.

Antilinear structures like a sesquilinear complex scalar product d are more complicated than linear ones. In general for a complex linear space

$\mathbb{U} \cong \mathbb{C}^n$, one has to consider the complex quartet[18] of associated vector spaces $\mathbb{U}$, $\mathbb{U}^T$, $\mathbb{U}^*$, $\mathbb{U}^{*T} \cong \mathbb{C}^n$, consisting of space, dual space, antispace, and dual antispace, respectively (Bourbaki, 1989a; Haft, 1997), to take care of the conjugations in a basic independent form. The canonical $\mathbb{C}$-conjugation defines canonical antilinear isomorphisms between antispaces $\mathbb{U} \cong \mathbb{U}^*$ and $\mathbb{U}^* \cong \mathbb{U}^{*T}$. With an additional vector space conjugation, i.e., an antilinear isomorphisms between duals, d: $\mathbb{U} \to \mathbb{U}^T$, $v \mapsto d(v, \cdot)$, one obtains linear isomorphism $\mathbb{U} \cong \mathbb{U}^{*T}$ and $\mathbb{U}^T \cong \mathbb{U}^*$.

There is a real 4-dimensional manifold (symmetric space) $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ of positive unitary groups in the general linear group, considered as a real 8-dimensional Lie group. With a reference basis, this manifold is parametrizable by all positive 2×2 matrices for the scalar products

$$d \mapsto \begin{pmatrix} d_0 + d_3 & d_1 - id_2 \\ d_1 + id_2 & d_0 - d_3 \end{pmatrix}$$

$$\succ 0 \Leftrightarrow \begin{cases} d = d^* \quad \text{and} \quad \operatorname{tr} d, \quad \det d > 0 \\ \text{i.e.,} \quad d_j \in \mathbb{R} \quad \text{and} \quad d_0, \quad d^2 = d_0^2 - \vec{d}^{\,2} > 0 \end{cases}$$

In analogy to $\alpha = \varepsilon(\alpha)\alpha$ for a positive number $\alpha > 0$, the positivity of the matrix d is expressible with its signature $\varepsilon(d) = \varepsilon(d_0)\vartheta(d^2)$,

$$d \succ 0 \Leftrightarrow d \neq 0 \qquad \text{and} \qquad d = \varepsilon(d_0)\vartheta(d^2)d$$

Besides the analogies, there are important differences between the real-orthogonal quotient structure of relativity and the complex-compact one proposed for the standard model: In contrast to the different dimensions of the spacetime and tetrad manifold in relativity for $\mathbb{M} \cong \mathbb{R}^4$,

$$4 = 1 + s = \dim_{\mathbb{R}}\mathcal{D} < \dim_{\mathbb{R}}\mathbf{GL}(\mathbb{R}^{1+s})/\mathbf{O}(1, s) = \begin{pmatrix} 2 + s \\ 2 \end{pmatrix} = 10$$

one has coinciding dimensions for 4-dimensional spacetime and the scalar product manifold for $\mathbb{U} \cong \mathbb{C}^2$

$$\dim_{\mathbb{R}}\mathcal{D} = \dim_{\mathbb{R}}\mathbf{GL}(\mathbb{C}^n)/\mathbf{U}(n) = n^2 = 4$$

Consequently, the symmetric space $\mathbf{D}(2)$ can be used (Saller, 1997b) as a model for the spacetime manifold

$$\mathcal{D} = \mathbf{D}(2) = \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) \cong \exp \mathbb{R}^4$$

With this interpretation, spacetime arises as the manifold of compact operations $\mathbf{U}(2)$ in general linear operations $\mathbf{GL}(\mathbb{C}^2)$.

[18] The complex quartet structure leads also to the fourfold concept 'particle creation, particle annihilation, antiparticle creation, and antiparticle annihilation.'

The full linear group $\mathbf{GL}(\mathbb{C}^2)$ is the direct product of its dilatation group $\mathbf{D}(\mathbf{1}_2) = \mathbf{1}_2 \exp \mathbb{R}$ and its unimodular group $\mathbf{UL}(2)$,

$$\mathbf{GL}(\mathbb{C}^2) = \mathbf{D}(\mathbf{1}_2) \times \mathbf{UL}(2)$$

$$\mathbf{D}(2) = \mathbf{GL}(\mathbb{C}^2)/\mathrm{U}(2) \cong \mathbf{D}(1) \times \mathbf{SD}(2)$$

The spacetime manifold $\mathbf{D}(2)$ involves as direct nonabelian factor the real 3-dimensional boost manifold

$$\mathbf{SD}(2) = \mathbf{UL}(2)/\mathrm{U}(2) \cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$$

The tangent spaces of the homogeneous space as the quotient of the corresponding Lie algebras

$$\log \mathbf{GL}(\mathbb{C}^2)/\log \mathbf{U}(2) \cong \mathbb{M} \cong \mathbb{R}^4$$

can be taken for the Minkowski translations carrying the irreducible $\mathbf{SL}(\mathbb{C}^2)$ representations $[1|1]$ of the adjoint group $\mathbf{GL}(\mathbb{C}^2)/\mathbf{GL}(\mathbb{C}) \cong \mathbf{SO}^+(1, 3)$. The Cartan representation of the spacetime translations $\mathbb{M}$ by the $\mathbf{U}(2)$ hermitian complex 2×2 matrices $x = x^*$ shows the local $\mathbf{U}(2)$ structure,

$$\mathbb{M} \cong \mathbf{D}(2) = \exp \mathbb{M}, \qquad \mathbb{R}^4 \cong \exp \mathbb{R}^4$$

$$x = x^* = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

$$d(x) = \exp x = \left(\cosh |\vec{x}| + \frac{\vec{\sigma}\vec{x}}{|\vec{x}|} \sinh |\vec{x}| \right) \exp x_0$$

$$= \begin{pmatrix} d_0(x) + d_3(x) & d_1(x) - id_2(x) \\ d_1(x) + id_2(x) & d_0(x) - d_3(x) \end{pmatrix}$$

In the special manifold factors $\mathbf{SL}_0(\mathbb{R}^4)/\mathbf{SO}^+(1, 3)$ (manifold of natural orders) and $\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$ (manifold of conjugations), the orthogonal stability group $\mathbf{SO}^+(1, 3)$ has a signature (1, 3) invariant Lorentz form g on the translations $\mathbb{M} \cong \mathbb{R}^4$, whereas the unitary group $\mathbf{SU}(2)$ has, in addition to an invariant scalar product d on $\mathbf{U} \cong \mathbb{C}^2$, an invariant antisymmetric bilinear form $\varepsilon(v, w) = -\varepsilon(w, v)$ ('spinor metric'). The $\mathbb{C}^2$-volume form ε is invariant also with respect to $\mathbf{SL}(\mathbb{C}^2)$; it leads to the bilinear symmetric orthochronous $\mathbf{SO}^+(1, 3)$-forms $g \cong \varepsilon \otimes \varepsilon^{-1}$. No $\mathbf{SL}_0(\mathbb{R}^4)$-invariant bilinear form exists on the translations $\mathbb{M}$.

7. SPACETIME AS BASIC FIELD QUANTIZATION

In analogy to the relativity tetrad h as basic representation of the real 10-dimensional metric manifold $\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3)$, a basic field ψ is introduced as fundamental representation for the real 4-dimensional manifold $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ of scalar products. It associates to each point of the real 4-dimensional spacetime $\mathcal{D} = \mathbf{D}(2)$, parametrizable with $d(x) = \exp x$ for $x \in \mathbb{R}^4$, a class representative

$$\psi\colon \quad \mathbf{D}(2) \to \mathbf{GL}(\mathbb{C}^2), \qquad d(x) \cong x \mapsto \psi(x)$$

With the basic field ψ, a complex vector space $\mathbb{U}(x) \cong \mathbb{C}^2$ at each spacetime point can be related to a reference space. $\psi^*(x)$ gives an isomorphism between the reference antispace $\mathbb{U}^*(0)$ and the antispace $\mathbb{U}^*(x)$,

$$\psi(x)\colon \quad \mathbb{U}(x) \to \mathbb{U}(0), \qquad \psi^{*-1}(x)\colon \quad \mathbb{U}^*(x) \to \mathbb{U}^*(0)$$

with the scalar products

$$\begin{array}{ccc} (\mathbb{U} \times \mathbb{U})(x) & \xrightarrow{d(x)} & \mathbb{C} \\ {\scriptstyle (\psi\times\psi)(x)}\big\downarrow & & \big\downarrow {\scriptstyle id_{\mathbb{C}}}, \\ (\mathbb{U} \times \mathbb{U})(0) & \xrightarrow[d(0)]{} & \mathbb{C} \end{array} \qquad d(x) = d(0) \circ (\psi \times \psi)(x)$$

Bases are given with $\alpha, A = 1, 2$:

$$\psi(x) \sim \psi^\alpha_A(x) \sim \psi^T(x), \qquad \psi^{*-1}(x) \sim \psi^{*\dot{A}}_\beta(x) = \delta_{\alpha\beta}\psi^{*\alpha}_{\dot{A}}(x)\delta^{A\dot{A}} \sim \psi^{*-1T}(x)$$

$$d(0) \sim \delta_{\alpha\beta}$$

$$d(x) \sim d_{AB}(x) = \delta_{\beta\alpha}\psi^{*\alpha}_{\dot{A}}\psi^\beta_B(x) \cong d^{\dot{A}}_B(x) = d_{AB}(x)\delta^{A\dot{A}} = \psi^{*\dot{A}}_\beta\psi^\beta_B(x)$$

The basic fields ψ and ψ^* transform under the two conjugated fundamental complex 2-dimensional $\mathbf{UL}(2)$ representations (left- and right-handed Weyl spinors), as usual denoted with undotted and dotted indices.

The $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ analogue to the flat spacetime expansion in general relativity $\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3)$ with the tetrad expansion $h = \mathbf{1}_4 + \ldots$ around a reference $\mathbf{O}(1, 3)$ requires an expansion of the external group $\mathbf{UL}(2)$ around a compact local reference group $\mathbf{U}(2)$. Such an expansion in the standard model is performed by the transition from the operation group representing fields for the dynamics to the tangent particle fields (in- and out-fields) involving the dramatic symmetry reduction mentioned above and requires the definition of a ground state and a reference system (spontaneous symmetry breakdown).

By an expansion of the coset representative ψ in flat spacetime $\mathbb{M} \cong \mathbb{R}^4$ with the standard model lepton fermion field $\mathbf{l}$

$$\psi(x) = \mathbf{l}(x) + \ldots, \qquad \psi^\alpha_A(x) = \mathbf{l}^\alpha_A(x) + \ldots$$

$$\psi^*(x) = \mathbf{l}^*(-x) + \ldots, \qquad \psi^{*\dot{A}}_\alpha(x) = \mathbf{l}^{*\dot{A}}_\alpha(-x) + \ldots$$

the spacetime defining scalar product can be related to the anticommutator quantization condition[19]

$$\log d(x) = \{\psi^*(x), \psi(x)\} = \{\mathbf{l}^*(-x), \mathbf{l}(x)\} + \ldots = x\varepsilon(x_0)\delta'(x^2) + \ldots$$
$$\log d^{\dot{A}}_{B}(x) = \{\psi^{*\dot{A}}_{\dot{\beta}}(x), \psi^{\beta}_{B}(x)\} = \{\mathbf{l}^{*\dot{A}}_{\dot{\alpha}}(-x), \mathbf{l}^{\alpha}_{B}(x)\} + \ldots = x^{\dot{A}}_{B}\varepsilon(x_0)\delta'(x^2) + \ldots$$

with

$$x = x^* \sim x^{\dot{A}}_{B} = (\sigma^j)^{\dot{A}}_{B} x_j = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

With the canonically quantized flat-space standard model fields alone a coset interpretation breaks down at this point. A quantization involving light-cone-supported distributions does not allow an interpretation as a spacetime-dependent scalar product $d(x)$. Additional nonparticle contributions (Saller, 1997a,b) can lead to an expansion for the basic field quantization without light-cone-supported distribution

$$\log d(x) = \{\psi^*(x), \psi(x)\} = x\varepsilon(x_0)\vartheta(x^2) + \ldots$$

The parametrization of the spacetime manifold **D**(2) is effected by the quantization of the basic field ψ.

8. THE SCALES FOR THE STANDARD MODEL

The representations of the scalar product manifold **D**(2) with real rank 2 as model for spacetime are characterized by two real invariants, an abelian dilatation invariant M and a simple 'boost'-invariant m:

$$\mathrm{rank}_{\mathbb{R}}\, \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) = \mathrm{rank}_{\mathbb{R}}\, \mathbf{D}(1) + \mathrm{rank}_{\mathbb{R}}\, \mathbf{SO}^+(1, 3)/\mathbf{SO}(3) = 2$$

$$d(x) \cong \exp M \begin{pmatrix} \exp m & 0 \\ 0 & \exp(-m) \end{pmatrix} = \psi(M, m) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \psi^*(M, m)$$

The two invariants, given in the Lie algebra structure by the abelian trace and the simple 'double trace'

[19] For the left-handed part of the massive lepton particle field one has the anticommutator

$$\{\mathbf{l}^*(0), \mathbf{l}(x)\} = \int \frac{d^4q}{8\pi^2} q\varepsilon(q_0)\delta(q^2 - M^2) \exp iqx$$
$$= x\varepsilon(x_0)\left[\delta'(x^2) - \frac{M^2}{4}\delta(x^2) + \frac{M^4}{16}\vartheta(x^2) + \cdots\right]$$

$$\psi(M, m) = \exp \frac{M}{2} \begin{pmatrix} e^{m/2} & 0 \\ 0 & e^{-m/2} \end{pmatrix},$$

$$\psi(m) = \frac{\psi(M, m)}{\left|\sqrt{\det \psi(M, m)}\right|} = \begin{pmatrix} e^{m/2} & 0 \\ 0 & e^{-m/2} \end{pmatrix}$$

$$M = \operatorname{tr} \log \psi(M, m),$$

$$m^2 = 2 \operatorname{tr} \log \psi(m) \circ \log \psi(m)$$

can be used (Saller, 1997b) as fundamental mass scale M and fundamental interaction range $1/m$ in the representations of the spacetime manifold $\mathbf{D}(2) \cong \mathbf{D}(1) \times \mathbf{SD}(2)$ by quantum fields.

9. HYPERISOPIN GAUGE FIELDS

The curvature in relativity $\mathcal{R}$: $(\mathbb{M} \otimes \mathbb{M}^T)(x) \to (\mathbb{M} \otimes \mathbb{M}^T)(0)$ relates to each other the Lorentz Lie algebras acting on the tangent spaces. The analogue for the standard model considers the tensor product $\mathbb{U} \otimes \mathbb{U}^* \cong \mathbb{C}^4$ for a scalar product space $\mathbb{U}$ with the represented group $\mathbf{U}(2)$ and its Lie algebra. The two real 4-dimensional subspaces $\{f = \pm f^* \big| f \in \mathbb{U} \otimes \mathbb{U}^*\}$ of the endomorphism $\mathbb{U} \otimes \mathbb{U}^* \cong \mathbb{R}^4 \oplus i\mathbb{R}^4$ are both stable under the action of $\mathbf{U}(2)$. The product representation of $\mathbf{U}(2)$ decomposes into a 3-dimensional representation, faithful for the adjoint group $\mathbf{SO}(3) \cong \mathbf{U}(2)/\mathbf{U}(1)$, and a 1-dimensional trivial one:

$$u \in \mathbf{U}(2): \quad u \otimes u: \quad \mathbb{U} \otimes \mathbb{U}^* \to \mathbb{U} \otimes \mathbb{U}^*$$
$$u \otimes u \cong \mathrm{id}_{\mathbb{U}} \oplus \mathcal{O}_3(u) \in \{1\} \oplus \mathbf{SO}(3)$$

$\mathbf{U}(2)$-gauge fields $\mathcal{G}$ associate to each spacetime point an isomorphism to a reference tensor space:

$$\mathcal{G}(x): \quad (\mathbb{U} \otimes \mathbb{U}^*)(x) \to (\mathbb{U} \otimes \mathbb{U}^*)(0)$$

$$\mathcal{G}(x) = (\psi \otimes \psi^*)(x) = \mathcal{A}(x) \oplus \mathcal{B}(x)$$

$$\mathcal{G}^{\dot{B}\alpha}_{A\beta}(x) = \psi^{\alpha}_{A}\psi^{*\dot{B}}_{\beta}(x) = (\sigma^j)^{\dot{B}}_{A}[\delta^{\alpha}_{\beta}\mathcal{A}_j(x) + \overline{\tau}^{\alpha}_{\beta}\overline{\mathcal{B}}_j(x)]$$

Therewith, the manifold of $(1 \oplus 3)$-decomposable 4-dimensional isospin $\mathbf{SO}(3)$ representations on the tensor product is considered in the orthochronous Lorentz group $\mathbf{SO}^+(1, 3) \cong \mathbf{UL}(2)/\mathbf{U}(1)$.

The hyperisospin $\mathbf{U}(2)$-gauge fields of the standard model might be taken as one term in the particle oriented flat spacetime approximation:

$$\mathcal{A}_j(x) = \tfrac{1}{4}\psi^{\alpha}_{A}\delta^{\beta}_{\alpha}(\overline{\sigma}_j)^{A}_{\dot{B}}\psi^{*\dot{B}}_{\beta}(x) = \mathbf{A}_j(x) + \ldots$$

$$\overline{\mathcal{B}}_j(x) = \tfrac{1}{4}\psi^{\alpha}_{A}\overline{\tau}^{\beta}_{\alpha}(\overline{\sigma}_j)^{A}_{\dot{B}}\psi^{*\dot{B}}_{\beta}(x) = \overline{\mathbf{B}}_j(x) + \ldots$$

In general, the standard model fields seem to be the particle-related and ground-state-respecting contributions in a flat-spacetime expansion for the more basic fields ψ, ψ^* which parametrize the U(2) operations in $\mathbf{UL}(2) \subset \mathbf{GL}(\mathbb{C}^2)$ acting on the tensor powers of the vector spaces $\mathbb{U}$, $\mathbb{U}^*$. This is done for the basic space $\mathbb{U}$ with faithful hyperisospin **U**(2) action by the standard lepton field $\psi = \mathbf{l} + \ldots$ and for the tensor space $\mathbb{U} \otimes \mathbb{U}^*$ with adjoint isospin group **U**(2)/**U**(1) action by the standard hypercharge and isospin gauge fields $\psi \otimes \psi^* = \mathbf{A} \oplus \mathbf{B} + \ldots$.

10. THE GRASSMANN ALGEBRA FOR SPACETIME

The local Grassmann algebra $\wedge\, \mathbb{M} \cong \mathbb{R}^{16}$ over the translations at each point of the spacetime manifold in relativity has as analogue the local Grassmann algebra over $\mathbb{U} \oplus \mathbb{U}^* \cong \mathbb{C}^4$ for the standard model. In contrast to the translations $\mathbb{M} \cong \mathbb{R}^4$, the vector space $\mathbb{U} \cong \mathbb{C}^2$ does not arise as a tangent space. The totally antisymmetric tensor powers $\wedge^N (\mathbb{U} \oplus \mathbb{U}^*)$ with Grassmann degree $N = 0, 1, 2, 3, 4$ carry all fundamental representations of hyperisospin **U**(2) and its quotient groups **U**(1) and **SO**(3). Their direct sum constitutes the complex Grassmann (exterior) algebra (Saller, 1993a,b) $\wedge (\mathbb{U} \oplus \mathbb{U}^*) = \mathbf{GRASS} \cong \mathbb{C}^{16}$ (Table X).

With the basic fermion field ψ the internal hyperisospin **U**(2) properties of a reference Grassmann algebra for $(\mathbb{U} \oplus \mathbb{U}^*)(0)$ are considered in the external Lorentz phase group **UL**(2) properties of a Grassmann algebra for $(\mathbb{U} \oplus \mathbb{U}^*)(x)$

$$\wedge(\psi \oplus \psi^*)(x): \quad \mathbf{GRASS}(x) \to \mathbf{GRASS}(0)$$

with isomorphism between corresponding vector subspaces with a corresponding external and internal representation structure (Table XI).

Table X. U(2) and **UL**(2) Properties of the Grassmann Algebra **GRASS**

N	Subspaces of $\wedge^N(\mathbb{U} \oplus \mathbb{U}^*) \cong \mathbb{C}^{(\text{po}\,{}^4_N)}$	Faithfully represented internal group	Faithfully represented external group
0	$\mathbb{C}$	{1}	{1}
1	$\mathbb{U}, \mathbb{U}^* \cong \mathbb{C}^2$	**U**(2)	**UL**(2)
2	$\mathbb{U} \otimes \mathbb{U}^* \cong \mathbb{C}^4$	**SO**(3)	$\mathbf{SO}^+(1, 3)$
	$\mathbb{U} \wedge \mathbb{U}, \mathbb{U}^* \wedge \mathbb{U}^* \cong \mathbb{C}$	**U**(1)	**U**(1)
3	$\mathbb{U} \otimes \mathbb{U}^* \wedge \mathbb{U}^* \cong \mathbb{C}^2$	**U**(2)	**UL**(2)
	$\mathbb{U} \wedge \mathbb{U} \otimes \mathbb{U}^* \cong \mathbb{C}^2$		
4	$(\mathbb{U} \wedge \mathbb{U}) \otimes (\mathbb{U} \wedge \mathbb{U})^* \cong \mathbb{C}$	{1}	{1}

Table XI. Quantum Numbers of the Basic Field Products

N	Basic field	$\mathbf{U}(2) = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SU}(2)$ $[y\|2T]$	$\mathbf{UL}(2) = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SL}(\mathbb{C}^2)$ $[c\|2J_L\|2J_r]$
0	$\mathrm{id}_{\mathbb{U}}$	$[0\|0]$	$[0\|0\|0]$
1	$\psi(x)$, $\psi^*(x)$	$[-\frac{1}{2}\|1]$, $[+\frac{1}{2}\|1]$	$[+\frac{1}{2}\|1\|0]$, $[-\frac{1}{2}\|0\|1]$
2	$(\psi \otimes \psi^*)(x)$	$[0\|0] \oplus [0\|2]$	$[0\|1\|1]$
	$(\psi \wedge \psi)(x)$, $(\psi \wedge \psi)^*(x)$	$[\mp 1\|0]$	$[\pm 1\|0\|0]$
3	$(\psi \otimes \psi^* \wedge \psi^*)(x)$	$[+\frac{1}{2}\|1]$	$[-\frac{1}{2}\|1\|0]$
	$(\psi \wedge \psi \otimes \psi^*)(x)$	$[-\frac{1}{2}\|]$	$[+\frac{1}{2}\|0\|1]$
4	$(\psi \wedge \psi) \otimes (\psi \wedge \psi)^*(x)$	$[0\|0]$	$[0\|0\|0]$

A basic field ψ, quantized with anticommutators, cannot imbed the $\mathbf{U}(1)$ properties of $\mathbb{U} \wedge \mathbb{U} \cong \mathbb{C}$ with Grassmann degree $N = 2$, since the scalar combination vanishes,

$$\psi \wedge \psi(x):\quad \psi^{\alpha}_{A}\varepsilon^{AB}\varepsilon_{\alpha\beta}\psi^{\beta}_{B}(x) = \tfrac{1}{2}\varepsilon^{AB}_{\varepsilon_{\alpha\beta}}\{\psi^{\alpha}_{A}(x), \psi^{\beta}_{B}(x)\} = 0$$

Only the combination leading to an $\mathbf{SU}(2)$-triplet is nontrivial

$$\psi \wedge \psi(x) \sim \psi^{\alpha}_{A}\varepsilon^{AB}\bar{\bar{\tau}}_{\alpha\beta}\psi^{\beta}_{B}(x), \qquad \bar{\bar{\tau}}_{\alpha\beta} = \varepsilon_{\alpha\gamma}\bar{\bar{\tau}}^{\gamma}_{\beta} = \bar{\bar{\tau}}_{\beta\alpha}$$

Therewith one has to consider four types of nontrivial fields—two fermionic fields with odd Grassmann degree 1 and 3 and two bosonic fields with even Grassmann degree 2 and 4. Only $N = 1, 2, 3$ characterize nontrivial symmetric spaces and representations of the nonabelian boost manifold (conjugation manifold) $\mathbf{UL}(2)/\mathbf{U}(2)$ (Table XII).

In addition to the $\mathbf{D}(1)$ grading with the natural number Grassmann degree $N \in \mathbb{N}$, a Grassmann algebra over a self-dual complex space $\mathbb{U} \oplus \mathbb{U}^* \cong \mathbb{C}^{2n}$ has a $\mathbf{U}(1)$ grading with $z \in \mathbb{Z}_{2n+1}$. The $\mathbf{U}(1)$ property defines the hypercharge and chirality $\mathbb{Z}_5$ grading with $y, c = z/2 = 0, \pm\frac{1}{2}, \pm 1$,

Table XII. $\mathbf{UL}(2)/\mathbf{U}(2)$ Representations by Basic Field Products

N	$z/2$	Basic field	Manifold representation
1	$\mp\frac{1}{2}$	$\psi(x)$, $\psi^*(x)$	$\mathbf{UL}(2)/\mathbf{U}(2)$
2	0	$(\psi \otimes \psi^*)(x)$	$\mathbf{SO}^+(1, 3)/\{1\} \oplus \mathbf{SO}(3)$
3	$\pm\frac{1}{2}$	$(\psi \otimes \psi^* \wedge \psi^*)(x)$	$\mathbf{UL}(2)/\mathbf{U}(2)$
		$(\psi \wedge \psi \otimes \psi^*)(x)$	
4	0	$(\psi \wedge \psi) \otimes (\psi \wedge \psi)^*(x)$	$\{1\}$

$$\mathbf{GRASS} = \bigoplus_{z=-2}^{2} \mathbb{U}^{(z)}, \quad \begin{cases} \mathbb{U}^{(0)} = \mathbb{C} \oplus [\mathbb{U} \otimes \mathbb{U}^*] \oplus [\mathbb{U} \wedge \mathbb{U} \otimes \mathbb{U}^* \wedge \mathbb{U}^*] \cong \mathbb{C}^6 \\ \mathbb{U}^{(1)} = \mathbb{U}^* \oplus [\mathbb{U} \otimes \mathbb{U}^* \wedge \mathbb{U}^*] \cong \mathbb{C}^4 \\ \mathbb{U}^{(-1)} = \mathbb{U} \oplus [\mathbb{U} \wedge \mathbb{U} \otimes \mathbb{U}^*] \cong \mathbb{C}^4 \\ \mathbb{U}^{(2)} = \mathbb{U}^* \wedge \mathbb{U}^* \cong \mathbb{C} \\ \mathbb{U}^{(-2)} = \mathbb{U} \wedge \mathbb{U} \cong \mathbb{C} \end{cases}$$

A basic theory for the symmetric space $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ has to use only the field ψ in analogy to the tetrad h for minimal relativity $\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3)$. The standard model is not basic in this sense. But at least the correspondence between the relevant basic field products and the effective particle oriented standard fields can be found.

11. EXTERNAL–INTERNAL COSETS IN THE LEPTON MODEL

The ‘colorless’ standard model, i.e., without quark and gluon fields, parametrizes all nontrivial external–internal or internal–external cosets, $G_{\text{ext}}/G_{\text{int}}$ and $G_{\text{int}}G_{\text{ext}}$, respectively, which are possible with the $\mathbf{UL}(2)$ and $\mathbf{U}(2)$ representations in the Grassmann algebra (Table XIII). One has to consider the possibilities to embed into each other the nontrivial external groups $\mathbf{UL}(2)$, $\mathbf{U}(1)$, and $\mathbf{SO}^+(1, 3)$ and the nontrivial internal groups $\mathbf{U}(2)$, $\mathbf{U}(1)$, and $\mathbf{SO}(3)$—in both directions.

Internal $\mathbf{U}(2)$ can be embedded only in external $\mathbf{UL}(2)$, done by the lepton isodoublet fields, as the flat space contribution for the basic fields ψ, ψ^*

$$\mathbf{UL}(2)/\mathbf{U}(2): \quad \begin{cases} \mathbf{l}(x): & \mathbb{U}(x) \to \mathbb{U}(0), \quad \mathbf{l}(x) \sim \mathbf{l}_A^\alpha(x) \\ \mathbf{l}^*(x): & \mathbb{U}^*(x) \to \mathbb{U}^*(0), \quad \mathbf{l}^*(x) \sim \mathbf{l}_\alpha^{*\dot{A}}(x) \end{cases}$$

Internal $\mathbf{SO}(3)$ can be embedded only in external $\mathbf{SO}^+(1, 3)$, done by the gauge fields, corresponding to the basic field $\psi \otimes \psi^*$,

$$\mathbf{SO}^+(1, 3)/\{1\} \oplus \mathbf{SO}(3): \quad \mathbf{A}(x) \oplus \mathbf{B}(x): \quad (\mathbb{U} \otimes \mathbb{U}^*)(x) \to (\mathbb{U} \otimes \mathbb{U}^*)(0)$$
$$\mathbf{A}(x) \oplus \mathbf{B}(x) \sim \mathbf{A}_j(x) + \overline{\mathbf{B}}_j(x)$$

Table XIII

	$\mathbf{UL}(2)$	$\mathbf{U}(1)$	$\mathbf{SO}^+(1, 3)$
$\mathbf{U}(2)$	$\mathbf{l}$	$\mathbf{H}$	$\times$
$\mathbf{U}(1)$	$\mathbf{e}$	—	$\times$
$\{1\} \oplus \mathbf{SO}(3)$	$\times$	$\times$	$\mathbf{A} \oplus \mathbf{B}$

The embedding of external **U**(1) in internal **U**(1) is trivial, **U**(1)/**U**(1) $\cong \{1\}$.

Internal **U**(2) can imbed only external **U**(1), done by the Higgs isodoublet fields

$$\mathbf{U}(2)/\mathbf{U}(1):\quad \begin{cases} \mathbf{H}(x): & (\mathbb{U} \wedge \mathbb{U})(x) \to \mathbb{U}(0), \qquad \mathbf{H}(x) \sim \mathbf{H}^{\alpha}(x) \\ \mathbf{H}^{*}(x): & (\mathbb{U} \wedge \mathbb{U})^{*}(x) \to \mathbb{U}^{*}(0), \quad \text{<ql}\mathbf{H}^{*}(x) \sim \mathbf{H}^{*}_{\dot{\alpha}}(x) \end{cases}$$

Internal **U**(1) can be embedded only in external **UL**(2), done by the lepton isosinglet fields

$$\mathbf{UL}(2)/\mathbf{U}(1):\quad \begin{cases} \mathbf{e}(x): & \mathbb{U}^{*}(x) \to (\mathbb{U} \wedge \mathbb{U})(0), \quad \mathbf{e}(x) \sim \mathbf{e}_{\dot{A}}(x) \\ \mathbf{e}^{*}(x): & \mathbb{U}(x) \to (\mathbb{U} \wedge \mathbb{U})^{*}(0), \quad \mathbf{e}^{*}(x) \sim \mathbf{e}^{*A}(x) \end{cases}$$

The fields in the diagonal, the 2 × 2 lepton fields **l** (isodoublet Lorentz doublet), and the 4 × 4 gauge fields **A** ⊕ **B** [**U**(2)-quartet Lorentz vector] connect spaces with equal dimensions.

The pair (**H**, **e**) in the skew-diagonal with the 2 × 1 Higgs fields **H** (isodoublet Lorentz scalar) and the 1 × 2 lepton fields **e** (isosinglet Lorentz doublet) come together as a 'doublet property swapping pair,'

$$[y\|2T] \circ [c\|2J_L|2J_R] = \begin{cases} [1\|0] \circ [-\frac{3}{2}\|1|0] & \text{for } \mathbf{e}^{*} \\ [-\frac{1}{2}\|1] \circ [1\|0|0] & \text{for } \mathbf{H} \\ [\frac{1}{2}\|1] \circ [-\frac{1}{2}\|1|0] & \text{for } \mathbf{e}^{*} \otimes \mathbf{H} \end{cases}$$

The internal **SU**(2) for the Higgs Lorentz singlet field **H** corresponds to the external **SU**(2) ⊂ **SL**($\mathbb{C}^2$) for the lepton isosinglet

$$\frac{\underline{\mathbf{U}(1) \times \mathbf{UL}(2)}}{\mathbf{U}(2) \times \mathbf{U}(1)} \cong \mathbf{UL}(2)/\mathbf{U}(2)$$

The **SU**(2)-swapping pair can arise from the isomorphisms for the tensors of Grassmann degree 3,

$$\chi(x):\quad (\mathbb{U} \otimes \mathbb{U}^{*} \wedge \mathbb{U}^{*})(x) \to (\mathbb{U} \otimes \mathbb{U}^{*} \wedge \mathbb{U}^{*})(0)$$

$$\chi(x) = (\psi \otimes \psi^{*} \wedge \psi^{*})(x) \sim \chi^{\alpha}_{A}(x) = \psi^{\beta}_{A}\bar{\bar{\tau}}^{\alpha}_{\beta}\psi^{*\dot{C}}_{\gamma}\varepsilon_{\dot{C}\dot{D}}\bar{\bar{\tau}}^{\gamma\delta}\psi^{*\dot{D}}_{\delta}(x)$$

as a particle oriented twofold factorization in the flat spacetime expansion

$$(\psi \otimes \psi^{*} \wedge \psi^{*})(x) = (\mathbf{e}^{*} \otimes \mathbf{H})(x) + \ldots,$$

$$\chi^{\alpha}_{A}(x) = \varepsilon_{AB}\mathbf{e}^{*B}\mathbf{H}^{\alpha}(x) + \ldots$$

$$(\psi \wedge \psi \otimes \psi^{*})(x) = (\mathbf{H}^{*} \otimes \mathbf{e})(x) + \ldots,$$

$$\chi^{*\dot{A}}_{\dot{\alpha}}(x) = \varepsilon^{\dot{A}\dot{B}}\mathbf{H}^{*}_{\dot{\alpha}}\mathbf{e}_{\dot{B}}(x) + \ldots$$

12. QUARK FIELDS AS GRASSMANN ROOTS

The main problem for an interpretation of the standard model in the framework of a basic $\mathbf{GL}(\mathbb{C}^2)/\mathrm{U}(2)$ coset structure are the colored fields, the quark fields **q**, **d**, **u**, and the gluon fields **G**. The only natural relation of $\mathbf{U}(2 \times 3)$ to $\mathbf{U}(2)$ seems to arise in the Grassmann algebra **GRASS** $\cong \mathbb{C}^{16}$ over $\mathbb{U} \oplus \mathbb{U}^* \cong \mathbb{C}^4$, which gives rise to two types of faithful $\mathbf{U}(2)$ representations with Grassmann degrees $N = 1$ and $N = 3$, which may reflect color singlet and color triplet properties, respectively. In analogy to the representation of $\mathbb{U} \otimes (\mathbb{U} \wedge \mathbb{U})^*$ with the Higgs lepton two-factor product $\mathbf{e}^* \otimes \mathbf{H}$, the quarks may arise from a parametrization with a three-factor product,

$$\mathbb{U} \otimes \mathbb{U}^* \wedge \mathbb{U}^*: \quad \begin{cases} \mathbf{U}(2) \otimes \mathbf{U}(2)) \wedge \mathbf{U}(2) \cong \mathbf{U}(2) \otimes \mathbf{U}(1) \cong \mathbf{U}(2) \\ \mathbf{UL}(2) \otimes \mathbf{UL}(2)) \wedge \mathbf{UL}(2) \cong \mathbf{UL}(2) \otimes \mathbf{U}(1) \cong \mathbf{UL}(2) \end{cases}$$

taking care of the $\mathbf{GL}(\mathbb{C}) = \mathbf{D}(1) \times \mathbf{U}(1)$ properties given by the two gradings of the Grassmann algebra.

Originally, the quarks were introduced as 'cubic foot' representations of the nucleons with color $\mathbf{SU}(3)$ as gauge group for the strong interactions. As seen in the standard model central correlation $\mathbb{I}(3) \cong \mathbf{SU}(3) \cap \mathbf{U}(\mathbf{1}_3)$ (Section 4), a color $\mathbf{SU}(3)$ property with nontrivial triality (Baird and Bieden-harn, 1964), i.e., an $\mathbf{SU}(3)$ representation $[C_1, C_2]$ with $C_1 - C_2 \neq 3\mathbb{Z}$, e.g., triplets [1, 0] or sextets [2, 0], not, however, octets [1, 1] or decuplets [3, 0], cannot be separated from a third integer hypercharge $\mathbf{U}(1)$ property.

The $\mathbf{U}(3)$ hypercharge color group can be considered to be the continuous phase generalization of the discrete cyclotomic root $\exp(2\pi i/3) \in \mathbb{I}(3)$ or, in general for $\mathbf{U}(N)$ with centr $\mathbf{U}(N) \cong \mathbb{I}(N)$,

$$k = 1, \ldots, N: \quad \begin{cases} \exp \dfrac{2\pi i k}{N} = \left[\exp \dfrac{2\pi i}{N}\right]^k \quad \text{with} \quad \left[\exp \dfrac{2\pi i k}{N}\right]^N = 1 \\ \overset{k}{\wedge}\, \mathbf{U}(N) = \left\{\overset{k}{\wedge}\, u \,\middle|\, u \in \mathbf{U}(N)\right\} \quad \text{with} \quad \overset{N}{\wedge}\left[\overset{k}{\wedge}\, \mathbf{U}(N)\right] \cong \mathbf{U}(1) \end{cases}$$

Any root $\exp(2\pi i k/N) \in \mathbb{I}(N)$ as power of the cyclic root $\exp(2\pi i/N)$ has its correspondence in the group which is defined by the $\mathbf{U}(N)$ representation $\wedge^k\mathbf{U}(N)$ on a complex $\binom{N}{k}$-dimensional space as kth Grassmann power of the cyclic defining representation with $k = 1$,

$$\mathbb{I}(N) = \sqrt[N]{\overline{1}} = \left\{ \exp \frac{2\pi i k}{N} \middle| k = 1, \ldots, N \right\}$$

$$\sqrt[N]{\mathbf{U}(1)} = \left\{ \overset{k}{\wedge} \mathbf{U}(N) \middle| k = 1, \ldots, N \right\}, \qquad \text{with} \quad \mathbf{U}(N) \cong \frac{\mathbf{U}(1) \times \mathbf{SU}(N)}{\mathbf{I}(N)}$$

Table XIV shows examples the second, third, and sixth Grassmann roots of $\mathbf{U}(1)$.

With Grassmann powers one can define the Nth Grassmann root of $\mathbf{U}(n)$ for relatively prime[20] naturals (n, N), e.g., for the standard model isospin-color relevant pair $(n, N) = (2, 3)$,

$$\sqrt[N]{\mathbf{U}(n)} = \left\{ \overset{k}{\wedge} \mathbf{U}(n \times N) \middle| k = 1, \ldots, N \right\}$$

$$\text{with} \quad \mathbf{U}(n \times N) = \frac{\mathbf{U}(1) \times \mathbf{SU}(n) \times \mathbf{SU}(N)}{\mathbf{I}(n) \times \mathbf{I}(N)}$$

The root allows the distribution of the $\mathbf{U}(1)$ phase in $\mathbf{U}(n)$ on $k \le N$ factors, as shown, e.g., for $(n, N) = (2, 3)$ in Table XV.

Table XIV

k	1	2				
$\sqrt[2]{\overline{1}}$	$e^{2\pi i/2}$	$e^{2\pi i/1}$				
$\sqrt[2]{\mathbf{U}(1)}$	$\mathbf{U}(2)$	$\mathbf{U}(1)$				
k	1	2	3			
$\sqrt[3]{\overline{1}}$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	$e^{2\pi i/1}$			
$\sqrt[3]{\mathbf{U}(1)}$	$\mathbf{U}(3)$	$\mathbf{U}(3)$	$\mathbf{U}(1)$			
k	1	2	3	4	5	6
$\sqrt[6]{\overline{1}}$	$e^{2\pi i/6}$	$e^{2\pi i/3}$	$e^{2\pi i/2}$	$e^{-2\pi i/3}$	$e^{-2\pi i/6}$	$e^{2\pi i/1}$
$\sqrt[6]{\mathbf{U}(1)}$	$\mathbf{U}(6)$	$\mathbf{U}(6)/\mathbf{I}(2)$	$\mathbf{U}(6)/\mathbf{I}(3)$	$\mathbf{U}(6)/\mathbf{I}(2)$	$\mathbf{U}(6)$	$\mathbf{U}(1)$

[20] The Grassmann root of $\mathbf{U}(1)$ for any natural number $m = 1, 2, \ldots$ is obtained by using its Sylow decomposition $m = p_1^{k1} \cdots p_r^{kr}$ in powers of primes

$$\sqrt[m]{\mathbf{U}(1)} = \sqrt[p_2^{k_2}]{\sqrt[p_2^{k_2}]{\cdots \sqrt[p_r^{k_r}]{\mathbf{U}(1)}}} = \mathbf{U}(p_1^{k1} \times \cdots \times p_r^{kr}) = \frac{\mathbf{U}(1) \times \mathbf{SU}(p_1^{k1}) \times \cdots \times \mathbf{SU}(p_r^{kr})}{\mathbb{I}(p_1^{k1}) \times \cdots \times \mathbb{I}(p_r^{kr})}$$

Table XV

k	1	2	3
$\sqrt[3]{\sqrt[2]{1}}$	$\exp(\pm 2\pi i/6)$	$\exp(\pm 2\pi i/3)$	$\exp(2\pi i/2)$
		1ab$\mathbf{U}(3)$	
$\sqrt[3]{\mathbf{U}(2)}$	$\mathbf{U}(2 \times 3)$		$\mathbf{U}(2)$

The quark fields as cubic Grassmann roots take care of the basic field products with Grassmann degree $N = 3$ in $\mathbb{U} \otimes \mathbb{U}^* \wedge \mathbb{U}^*, \mathbb{U} \wedge \mathbb{U} \otimes \mathbb{U}^* \cong \mathbb{C}^2$. The quark isodoublet field $\mathbf{q}$ parametrizes the $k = 1$ member of the internal $\mathbf{U}(2)$ roots $\sqrt[3]{\mathbf{U}(2)}$ with $\mathbf{U}(2 \times 3)$ degrees of freedom, the two quark isosinglets $\mathbf{d}$, $\mathbf{u}$ parametrize the $k = 2$ member of $\sqrt[3]{\mathbf{U}(2)}$ with $\mathbf{U}(3)$ degrees of freedom

$$\sqrt[3]{\psi \otimes \psi^* \wedge \psi^*} = \begin{cases} \mathbf{q} + \ldots, & k = 1 \\ \mathbf{q} \wedge \mathbf{q} \oplus \mathbf{d} \wedge \mathbf{u} + \ldots, & k = 2 \\ \mathbf{q} \wedge \mathbf{q} \wedge \mathbf{q} \oplus \mathbf{q} \wedge \mathbf{d} \wedge \mathbf{u} + \ldots, & k = 3 \end{cases}$$

$$\sim \begin{cases} \mathbf{q}_A^{\alpha c} + \ldots \\ \varepsilon_{c_1 c_2 c_3} \varepsilon_{\alpha_2 \alpha_3} (\mathbf{q}_{A_2}^{\alpha_2 c_2} \varepsilon^{A_2 A_3} \mathbf{q}_{A_3}^{\alpha_3 c_3} + \mathbf{d}_{A_2}^{\alpha_2 c_2} \varepsilon^{\dot{A}_2 \dot{A}_3} \mathbf{u}_{A_3}^{\alpha_3 c_3}) + \ldots \\ \varepsilon_{c_1 c_2 c_3} \varepsilon_{\alpha_2 \alpha_3} \mathbf{q}_A^{\alpha c_1} (\mathbf{q}_{A_2}^{\alpha_2 c_2} \varepsilon^{A_2 A_3} \mathbf{q}_{A_3}^{\alpha_3 c_3} + \mathbf{d}_{A_2}^{\alpha_2 c_2} \varepsilon^{\dot{A}_2 \dot{A}_3} \mathbf{u}_{A_3}^{\alpha_3 c_3}) + \ldots \end{cases}$$

which is written with the representations

$$\sqrt[3]{[\tfrac{1}{2} \| 1]} = \begin{cases} [\tfrac{1}{6} \| 1; 1, 0], & k = 1 \\ [\tfrac{1}{3} \| 0; 0, 1], & k = 2, \quad \tfrac{1}{3} = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{2}{3} - \tfrac{1}{3} \\ [\tfrac{1}{2} \| 1; 0, 0], & k = 3 \end{cases}$$

If, for an effective linearization of the basic $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ coset structure as realized with the Grassmann algebra **GRASS** $\cong \mathbb{C}^{16}$, the basic internal operation group $\mathbf{U}(2)$ is extended by a cubic Grassmann root to $\mathbf{U}(2 \times 3)$, one has to provide also for a gauge field for the additional local $\mathbf{U}(2 \times 3)/\mathbf{U}(2) \cong \mathbf{SU}(3)/\mathbb{I}(3)$ operations. This is done in the standard model with the gluon fields $\mathbf{G}(x)$.

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